

A SHARP STABILITY ESTIMATE IN TENSOR TOMOGRAPHY

PLAMEN STEFANOV

1. INTRODUCTION

Let (M, g) be a compact Riemannian manifold with boundary. The geodesic ray transform I of symmetric 2-tensor fields f is given by

$$(1) \quad If(\gamma) = \int f_{ij}(\gamma(s)) \dot{\gamma}^i(s) \dot{\gamma}^j(s) ds,$$

where γ runs over the set of all geodesics with endpoints on ∂M . All *potential fields* dv given by $(dv)_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)$ with $v = 0$ on ∂M belong to the kernel of I . The ray transform I is called *s-injective* if this is the only obstruction to injectivity, i.e., if $If = 0$ implies that f is potential. S-injectivity can only hold under certain assumptions on (M, g) . A natural conjecture is that it holds on *simple* manifolds, see the definition below. So far it is known to be true for some classes of simple manifolds only, including generic simple manifolds, see [11, 13, 4, 16].

In the cases where s-injectivity is known, there is also a stability estimate that is not sharp. In [11], it is of conditional type with a loss of a derivative, see (2) below. In [16], the estimate is not of conditional type but there is still a loss of a derivative, see (3) below. On the other hand, if f is a function, or an 1-tensor (an 1-form), there is a sharp estimate, see [15]. The purpose of this paper is to prove a sharp estimate for the ray transform of 2-tensors.

The geodesic ray transform is a linearization of the boundary distance function and plays an important role in the inverse kinematic problem (known also as boundary or lens rigidity), see e.g., [11, 15, 18, 17] and the references there. There, one wants to recover (M, g) given the distance function on $\partial M \times \partial M$ or the scattering relation $\sigma : (x, \xi) \mapsto (y, \eta)$ that maps a given $x \in \partial M$ and a given incident direction ξ to the exit point y and the exit direction η of the geodesic issued from (x, ξ) .

2. MAIN RESULTS

Definition 1. *We say that a compact Riemannian manifold (M, g) with boundary is simple if*

- (a) *The boundary ∂M is strictly convex, i.e., $\langle \nabla_\xi \nu, \xi \rangle > 0$ for each $\xi \in T_x(\partial M)$ where ν is the unit outward normal to the boundary.*
- (b) *The map $\exp_x : \exp_x^{-1} M \rightarrow M$ is a diffeomorphism for each $x \in M$.*

Partly supported by NSF.

Condition (b) implies that each pair of points is connected by a unique geodesic depending smoothly on the endpoints. It also implies that M is diffeomorphic to a ball, so we can work in one fixed chart only. We will fix M fixed, and choose different metrics on it. If (M, g) is simple, then we call g simple.

It is known [11], see also [16], that each symmetric 2-tensor field $f \in L^2(M)$ admits an orthogonal decomposition

$$f = f^s + dv,$$

where v is 1-form in $H_0^1(M)$ (vanishing on ∂M), and f is divergence free, i.e., $\delta f = 0$, where $(\delta f)_j = \nabla^i f_{ij}$, and ∇ is the covariant derivative. The 1-form v solves $\delta dv = \delta f$, $v|_{\partial M} = 0$. The latter is an elliptic system and the Dirichlet boundary condition is a regular one for it, see [11, 16]. S-injectivity then is equivalent to the following: $If = 0$ implies $f^s = 0$.

Set

$$\partial_{\pm} SM := \{(x, \omega) \in TM; x \in \partial M, |\omega| = 1, \pm \langle \omega, \nu \rangle > 0\},$$

where $\nu(x)$ is the outer unit normal to ∂M (normal w.r.t. g , of course). Here and in what follows, we denote by $\langle \omega, \nu \rangle$ the inner product of the vectors ω, ν , and $|\omega|$ is meant w.r.t. g . Let $\gamma_{x, \omega}(t)$ be the (unit speed) geodesic through (x, ω) , defined on its maximal interval contained in $[0, \infty)$. One can then parametrize all maximal (directed) geodesics in M by points on $\partial_{-} SM$, and with some abuse of notation we denote $If(x, \omega) = If(\gamma_{x, \omega})$.

One of the methods to study s-injectivity of I is the energy estimates method that goes back to Mukhometov [8, 6, 7], see also [2], where, for simple manifolds, injectivity of I acting on functions f is proved. S-injectivity (injectivity up to $d\phi$, where $\phi = 0$ on ∂M) for 1-forms f is established in [1] by a modification of the same method. The case of 2-tensors is harder. Using the so-called Pestov identity, Sharafutdinov [11] showed that I is s-injective under an explicit a priori bound on the positive part of the curvature of g , that in particular implies simplicity of g but it is not equivalent to it. This generalized earlier results on negatively curved manifolds [9]. The Pestov-Sharafutdinov approach implies the following stability estimate [11] under the small curvature assumption:

$$(2) \quad \|f^s\|_{L^2(\Omega)}^2 \leq C \left(\|j_{\nu} f|_{\partial M}\|_{L^2(\partial M)} \|If\|_{L^2(\partial_{-} SM)} + \|If\|_{H^1(\partial_{-} SM)}^2 \right), \quad \forall f \in H^1(M),$$

where $(j_{\nu} f)_i = f_{ij} \nu^j$. The Sobolev spaces on the r.h.s. are taken with respect to the induced measure $d\sigma(x, \omega)$. In semigeodesic local coordinates $x = (x', x^n)$, the latter is given by $d\sigma(x, \omega) = (\det g)^{1/2} dx^1 \dots dx^{n-1} (\det g)^{1/2} d\sigma_x(\omega)$, where $d\sigma_x(\omega) = \frac{1}{|\xi|} \left| \sum_{j=1}^n (-1)^{j-1} \xi^j d\xi^1 \wedge \dots \wedge \widehat{d\xi^j} \wedge \dots \wedge d\xi^n \right|$.

This estimate is of conditional type — while it implies s-injectivity (under the curvature condition), it says that f^s is small if If is small *and* we have an a priori bound on f . Moreover, there is a loss of one derivative: the r.h.s. is finite under the condition that $f \in H^1$, while in the l.h.s., we have only the L^2 norm of f^s .

In [15, 16, 18], the author and G. Uhlmann studied this problem from microlocal point of view. Introduce the following measure $d\mu(x, \omega) = |\langle \nu, \omega \rangle| d\sigma(x, \omega)$ on $\partial_{-} SM$. It is easy to show that $I : L^2(M) \rightarrow L^2(\partial_{-} SM, d\mu)$ is bounded [11] (the first space is a space of tensors, actually). The *normal operator* $N = I^* I$ is well defined on $L^2(M)$ then. We extend slightly M to a larger manifold with boundary M_1 that is still simple so that $M_1 \supsetneq M$. We extend tensors defined in M as zero to M_1 . Then $I : L^2(M_1) \rightarrow L^2(\partial_{-} SM_1, d\mu)$, and one can define

I^*I related to M_1 that a priori is different from N . On the other hand, it is easy to see that when restricted to tensors supported in M , it coincides with N . Then $N : L^2(M) \rightarrow L^2(M_1)$ is s-injective if and only if I is s-injective.

The main results in [15, 16] concerning the tensor tomography problem are the following. The operator I is s-injective for real analytic simple metrics. Moreover, the set of simple C^k metrics, where $k \gg 1$ is fixed, for which I is s-injective, is open and dense in C^k . Therefore, we get s-injectivity for a generic set of simple metrics. Next, for any simple g for which I is s-injective, one has the stability estimate

$$(3) \quad \|f^s\|_{L^2(M)} \leq C \|Nf\|_{\tilde{H}^2(M_1)}, \quad \forall f \in H^1(M).$$

Here $\tilde{H}^2(M)$ is defined as follows. To the usual $H^1(M_1)$ norm, we add a term of the kind

$$\sum_{\alpha < n} \|\partial_{x^\alpha} \nabla f\|_{L^2(U)} + \|x^n \partial_{x^n} \nabla f\|_{L^2(U)},$$

where (x', x^n) are semigeodesic local coordinates near ∂M , and U is any fixed neighborhood of ∂M . The constant C in (3) can be chosen locally uniform for $g \in C^k$.

A natural conjecture is that I is injective for all simple metrics. This is still an open problem. For any simple metric however, we have an estimate of the kind (3) plus the term $\|Kf\|_{L^2(M)}$, where K is a smoothing operator. In Proposition 1 below we prove a sharper estimate of this kind. It is also known that the solenoidal tensors on the kernel of I form a finitely dimensional space of smooth tensors [12, 15, 16, 3, 10]. This also follows directly from the analysis below, since the inversion problem is reduced to a Fredholm one.

Estimate (3) is not of conditional type anymore but there is still a loss of one derivative. Indeed, N is a Ψ DO of order -1 , and the natural norm on the r.h.s. of (3) would be the $H^1(M_1)$ one. Our main result shows that this is the case, indeed.

Theorem 1. *Let $g \in C^k(M)$, $k \gg 1$, be a simple metric on M , and assume that I is s-injective.*

(a) *Then*

$$(4) \quad \|f^s\|_{L^2(M)}/C \leq \|Nf\|_{H^1(M_1)} \leq C \|f^s\|_{L^2(M)}$$

with some $C > 0$.

(b) *The constant C can be chosen uniformly under a small $C^3(M)$ perturbation of g .*

As pointed out above, the s-injectivity assumption is generically true for simple metrics, and holds in particular for metrics close enough to analytic ones [16] or for metrics with an explicit bound on the curvature [11].

The new ingredient of the proof is the use of Korn's inequality [19, Corollary 5.12.3], see (13).

In [17], the author and G. Uhlmann considered manifolds that are not simple, with possible conjugate points, and studied the question of the s-injectivity on I known on a subset Γ of geodesics. The basic assumption is that none of the geodesics in Γ has conjugate points, and the conormal bundles of all $\gamma \in \Gamma$ cover T^*M . Under that assumption, results about s-injectivity for generic simple metrics, including analytic ones are obtained. A stability estimate of the kind (3) is also proven there, where N is modified via a smooth cut-off that

restricts the geodesics to Γ . Without going into detail, we will only mention that Theorem 1 generalizes to that case, i.e., the stability estimate in [17] can be written in the form (4) as well.

Theorem 1 allows us to reduce the smoothness requirement in the generic result in [16].

Corollary 1. *There exists a dense open set of simple metrics in $C^3(M)$ so that the corresponding ray transform I is s-injective (and (4) holds).*

Note that we are not claiming that the set of all $C^3(M)$ simple metrics with an s-injective I is open. Our success with proving estimate (4) that implies the openness depends on our ability to show that the problem can be reduced to a Fredholm one. We do this by constructing a parametrix, and this requires certain number k of derivatives of g , at least $k = 2n + 1$. Once we have (4), we use the singular operator theory to perturb (4) near any $g_0 \in C^k$ with an s-injective I , by C^3 perturbations.

3. PROOFS.

3.1. Proof of Theorem 1(a). We start with recalling some facts from [15, 16]. We show first that

$$(Nf)^{i'j'}(x) = 2 \int_{S_x M} \omega^{i'} \omega^{j'} \int_0^\infty f_{ij}(\gamma_{x,\omega}(t)) \dot{\gamma}_{x,\omega}^i(t) \dot{\gamma}_{x,\omega}^j(t) dt d\sigma_x(\omega),$$

where f is supported in M , and we work in M_1 . Performing a change of variables, we get the integral representation

$$(5) \quad (Nf)_{kl}(x) = \frac{2}{\sqrt{\det g(x)}} \int \frac{f^{ij}(y)}{\rho(x,y)^{n-1}} \frac{\partial \rho}{\partial y^i} \frac{\partial \rho}{\partial y^j} \frac{\partial \rho}{\partial x^k} \frac{\partial \rho}{\partial x^l} \Big|_{\det} \frac{\partial^2(\rho^2/2)}{\partial x \partial y} \Big| dy, \quad x \in M_1,$$

where ρ is the distance function. This form of N show that N is a Ψ DO of order -1 on the interior of M_1 . Its principal symbol is, see [16, 14],

$$(6) \quad \sigma_p(N)^{ijkl}(x, \xi) = 2\pi \int_{S_x M_1} \omega^i \omega^j \omega^k \omega^l \delta(\xi \cdot \omega) d\sigma_x(\omega),$$

where $\xi \cdot \omega = \xi_i \omega^i$. This formula generalizes in an obvious way to tensors of any order. It follows now easily that N is elliptic on tensors satisfying $\xi^i f_{ij} = 0$ (solenoidal tensors in the Fourier representation), and vanishes on tensors of the type $\frac{1}{2}(\xi_i v_j + \xi_j v_i)$ (potential tensors in the Fourier representation). This fact allows us to construct a first order Ψ DO Q so that for any $f \in L^2(M)$,

$$(7) \quad QNf = f_{M_1}^s + Kf$$

in M_1 , where $f_{M_1}^s$ is the solenoidal projection of f (extended as zero outside M) in M_1 , and K is a compact operator. We can assume that the kernel of Q has a support close enough to the diagonal. The need to work in M_1 is due to the fact that we can use the (standard) Ψ DO calculus in an open set only. For more details, we refer to [15, 16]. Note that this construction needs only a finitely smooth metric $g \in C^k(M)$, $k \gg 1$, that we extend to M_1 . If we want K to be infinitely smoothing, then we need $g \in C^\infty(M)$.

The next step is to construct f^s , given $f_{M_1}^s$. This can be done in an explicit way as follows. Note that

$$(8) \quad f_{M_1}^s = Ef^s - dw \quad \text{in } M_1,$$

where E is the extension as zero to $M_1 \setminus M$, and w solves the elliptic system

$$(9) \quad \delta dw = \delta Ef^s, \quad w|_{\partial M_1} = 0.$$

The distribution δEf^s is supported on ∂M , and the solution w exists in $H_0^1(M_1)$, see [16] and Lemma 1 below. In particular, $w|_{\partial M} \in H^{1/2}(\partial M)$ is well-defined. If we know $w|_{\partial M} \in H^{1/2}(\partial M)$, we can recover w in M because $\delta dw = 0$ in the interior of M , by (9). If we recover w in M , we recover f^s as well, in terms of $f_{M_1}^s$, by (8). Our goal therefore is to recover $w|_{\partial M}$ first.

We first determine w in $M_1 \setminus M$, up to a smoothing term, by the relation

$$(10) \quad f_{M_1}^s = -dw \quad \text{in } M_1 \setminus M,$$

see (8). Since $w = 0$ on ∂M_1 , we can integrate the identity

$$(11) \quad \frac{d}{dt} w_i(\gamma) \dot{\gamma}^i = [dw(\gamma)]_{ij} \dot{\gamma}^i \dot{\gamma}^j$$

along geodesics in $M_1 \setminus M$ connecting points on ∂M_1 and ∂M to recover w on ∂M . Let $\tau_+(x, \xi) > 0$ be characterized by $\gamma_{x, \xi}(t) \in \partial M_1$ for $t = \tau_+(x, \xi)$. Then we get

$$w_i(x) \xi^i = \int_0^{\tau_+(x, \xi)} [f_{M_1}^s]_{ij}(\gamma_{x, \xi}(t)) \dot{\gamma}_{x, \xi}^i(t) \dot{\gamma}_{x, \xi}^j(t) dt,$$

for any (x, ξ) so that $\{\gamma_{x, \xi}(t), 0 \leq t \leq \tau_+(x, \xi)\}$ does not intersect M . That also implies easily the following non-sharp estimate

$$(12) \quad \|w\|_{L^2(M_1 \setminus M)} \leq C \|dw\|_{L^2(M)} \leq C \|f_{M_1}^s\|_{L^2(M_1 \setminus M)},$$

see also [11] for the first inequality. We refer to [15, 16] for more detail. This approach provides also a constructive way to reduce the problem to a Fredholm one. For the proof of the theorem however, this is not needed. The new ingredient in this work is that we apply Korn's inequality [19, Corollary 5.12.3],

$$(13) \quad \|w\|_{H^1(M_1 \setminus M)} \leq C (\|dw\|_{L^2(M_1 \setminus M)} + \|w\|_{L^2(M_1 \setminus M)}).$$

This inequality is a consequence of the fact that the Neumann boundary conditions for δd are regular ones. Apply the trace theorem, (13), (12), and (10) to get

$$\|w\|_{H^{1/2}(\partial M)} \leq C \|w\|_{H^1(M_1 \setminus M)} \leq C' \|f_{M_1}^s\|_{L^2(M_1 \setminus M)}.$$

Now, since w solves the elliptic PDE $\delta dw = 0$ in M_1 , we get

$$(14) \quad \|w\|_{H^1(M)} \leq C \|f_{M_1}^s\|_{L^2(M_1 \setminus M)},$$

see Lemma 1 below. This, together with (8) yields,

$$(15) \quad \|f^s\|_{L^2(M)} \leq \|f_{M_1}^s\|_{L^2(M)} + C \|f_{M_1}^s\|_{L^2(M_1 \setminus M)} \leq C (\|Nf\|_{H^1(M_1)} + \|Kf\|_{L^2(M_1)}).$$

It is worth noting that without the a priori s-injectivity assumption, we got the following.

Proposition 1. *For any $l > 0$, there exists $k > 0$ so that for any simple metric $g \in C^k(M)$,*

$$\|f^s\|_{L^2(M)} \leq C \left(\|Nf\|_{H^1(M_1)} + \|f\|_{H^{-s}(M)} \right), \quad \forall f \in L^2(M).$$

Estimate (15) (or Proposition 1), together with [19, Proposition 5.3.1], implies that if I , and therefore, N is s-injective, then there is an estimate as above, with a different C , with the last term missing. This completes the proof of Theorem 1(a).

We return to the elliptic regularity estimate (14). If δd is replaced by the Laplace operator, then (14) follows from [19, Theorem 5.1.3]. If we raise the Sobolev regularity everywhere in (17) below by 1, this just follows from the fact that the Dirichlet conditions are regular for δb . In our case, we follow the proof of [19, Theorem 5.1.3] to get the following.

Lemma 1. *Let $u \in H^{-1}(M)$, $\alpha \in H^{1/2}(\partial M)$ be 1-forms. Then the boundary value problem*

$$(16) \quad \delta dw = u \quad \text{in } M, \quad w|_{\partial M} = \alpha$$

has a unique solution $w \in H^1(M)$, and the following estimate holds

$$(17) \quad \|w\|_{H^1(M)} \leq C \left(\|u\|_{H^{-1}(M)} + \|\alpha\|_{H^{1/2}(\partial M)} \right)$$

Proof. By a standard argument, first extend $\alpha \in H^{1/2}(\partial M)$ to $\tilde{\alpha} \in H^1(M)$ by means of a fixed bounded extension operator, and then study $u - \tilde{\alpha}$ that satisfies homogeneous boundary conditions. This shows that we can assume that $\alpha = 0$, then the boundary condition is equivalent to $w \in H_0^1(M)$.

Note first that $\|w\|_{H^1(M)}$ and $\|dw\|_{L^2(M)}$ are equivalent norms on $H_0^1(M)$, by (13) and the Poincaré type of inequality for dw , see the first inequality in (12). The existence part of the theorem in the case $\alpha = 0$ then follows as in [[19, Proposition 5.1.1].

To prove the stability estimate, given $w \in H_0^1(M)$, integrate by parts to get

$$\|w\|_{H^1}^2 / C \leq \|dw\|_{L^2}^2 = -(\delta dw, w) \leq \|\delta dw\|_{H^{-1}} \|w\|_{H^1}.$$

That implies (16) when $\alpha = 0$, and completes the proof of the lemma. \square

Remark 1. Lemma 1 in particular justifies the solenoidal-potential decomposition of tensors f with L^2 only regularity, see also [16]. Then $f = f^s + dv$ with $v \in H_0^1$ solving $\delta dv = \delta f$.

3.2. Proof of Theorem 1(b) and Corollary 1. To prove Corollary 1, we define the set \mathcal{G} of simple metrics as follows: near any real analytic simple g , we choose a small enough neighborhood in the $C^3(M)$ topology, so that (4) still holds. Clearly, this set is open and dense. It remains to prove that this can be done, which is the statement of Theorem 1(b).

We will prove a bit more. Fix a simple $g_0 \in C^3(M)$ (not necessarily real analytic) and assume that (4) holds; in particular, the corresponding ray transform I is s-injective. We will show that there exists $0 < \epsilon \ll 1$ so that for any other g with $\|g - g_0\|_{C^3(M)} < \epsilon$, (4) still holds with possibly a different constant $C > 0$, independent on g_0 and ϵ .

We will apply first [5, Proposition 4]. There, the weighted ray transform

$$I_w f(\gamma) = \int w(\gamma(s), \dot{\gamma}(s)) f(\gamma(s)) ds$$

of functions is studied. The estimate in [5, Proposition 4] compares two such transforms with different weights and different metrics (actually, we study more general families of curves

in [5]). In our context, we apply [5, Proposition 4] to each $f_{ij}\dot{\gamma}^i\dot{\gamma}^j$, before summing up by treating $\dot{\gamma}^i\dot{\gamma}^j$ as a weight. All we need to show is that the generators of the geodesic flows related to g_0 and g are $O(\epsilon)$ close in C^2 . This follows from our assumption $\|g - g_0\|_{C^3(M)} < \epsilon$. Then we get

$$(18) \quad \|(N_g - N_{g_0})f\|_{H^1(M_1)} \leq C\epsilon\|f\|_{L^2(M)}.$$

To perturb the l.h.s. of (4), we need to compare the solenoidal projections $f_{g_0}^s$ and f_g^s of f related to g_0 and g , respectively. Recall that $f^s = f - d(\delta d)_D^{-1}\delta f$, where $(\delta d)_D$ is the Dirichlet realization of the elliptic operator δd . By [16, Lemma 1], $(\delta d)_D^{-1} : H^{-1}(M) \rightarrow H_0^1(M)$ depends continuously on $g \in C^1(M)$. The same is true for d and δ in the corresponding spaces. Therefore,

$$(19) \quad \|f_g^s - f_{g_0}^s\|_{L^2(M)} \leq C_0\epsilon\|f\|_{L^2(M)}.$$

It is enough to prove (4) for f solenoidal, w.r.t. g , so, let us assume that. Then by (18), and our assumption that (4) holds for g_0 ,

$$\|f_{g_0}^s\|_{L^2(M)} \leq C\|N_{g_0}f\|_{H^1(M_1)} \leq C\epsilon\|f\|_{L^2(M)} + C\|N_gf\|_{H^1(M_1)}.$$

Apply (19), where $f_g^s = f$ to get

$$(1 - C_0\epsilon)\|f\|_{L^2(M)} \leq C\epsilon\|f\|_{L^2(M)} + C\|N_gf\|_{H^1(M_1)}.$$

Therefore, if $\epsilon \ll 1$, we still have (4). This completes the proof of Theorem 1(b). Now, Corollary 1 follows immediately.

REFERENCES

- [1] Y. E. Anikonov and V. G. Romanov. On uniqueness of determination of a form of first degree by its integrals along geodesics. *J. Inverse Ill-Posed Probl.*, 5(6):487–490 (1998), 1997.
- [2] I. N. Bernstein and M. L. Gerver. A problem of integral geometry for a family of geodesics and an inverse kinematic seismics problem. *Dokl. Akad. Nauk SSSR*, 243(2):302–305, 1978.
- [3] E. Chappa. On the characterization of the kernel of the geodesic X-ray transform. *Trans. Amer. Math. Soc.*, 358(11):4793–4807 (electronic), 2006.
- [4] N. S. Dairbekov. Integral geometry problem for nontrapping manifolds. *Inverse Problems*, 22(2):431–445, 2006.
- [5] B. Frigyik, P. Stefanov, and G. Uhlmann. The X-ray transform for a generic family of curves and weights. *J. Geom. Anal.*, 18(1):81–97, 2008.
- [6] R. G. Muhometov. The reconstruction problem of a two-dimensional Riemannian metric, and integral geometry. *Dokl. Akad. Nauk SSSR*, 232(1):32–35, 1977.
- [7] R. G. Muhometov. On a problem of reconstructing Riemannian metrics. *Sibirsk. Mat. Zh.*, 22(3):119–135, 237, 1981.
- [8] R. G. Mukhometov. On the problem of integral geometry (Russian). *Math. problems of geophysics, Akad. Nauk SSSR, Sibirsk., Otdel., Vychisl., Tsent, Novosibirsk*, 6(2):212–242, 1975.
- [9] L. N. Pestov and V. A. Sharafutdinov. Integral geometry of tensor fields on a manifold of negative curvature. *Sibirsk. Mat. Zh.*, 29(3):114–130, 221, 1988.
- [10] V. Sharafutdinov, M. Skokan, and G. Uhlmann. Regularity of ghosts in tensor tomography. *J. Geom. Anal.*, 15(3):499–542, 2005.
- [11] V. A. Sharafutdinov. *Integral geometry of tensor fields*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.
- [12] V. A. Sharafutdinov. A finiteness theorem for the ray transform on a Riemannian manifold. *Dokl. Akad. Nauk*, 355(2):167–169, 1997.

- [13] V. A. Sharafutdinov. Ray transform on Riemannian manifolds, lecture notes, UW–Seattle. available at: <http://www.ima.umn.edu/talks/workshops/7-16-27.2001/sharafutdinov/>, 1999.
- [14] P. Stefanov. Microlocal approach to tensor tomography and boundary and lens rigidity. *Serdica Math. J.*, 34(1):67–112, 2008.
- [15] P. Stefanov and G. Uhlmann. Stability estimates for the X-ray transform of tensor fields and boundary rigidity. *Duke Math. J.*, 123(3):445–467, 2004.
- [16] P. Stefanov and G. Uhlmann. Boundary rigidity and stability for generic simple metrics. *J. Amer. Math. Soc.*, 18(4):975–1003 (electronic), 2005.
- [17] P. Stefanov and G. Uhlmann. Local lens rigidity with incomplete data for a class of non-simple riemannian manifolds. *submitted*, 2007.
- [18] P. Stefanov and G. Uhlmann. Integral geometry of tensor fields on a class of non-simple riemannian manifolds. *Amer. J. Math.*, 130(1):239–268, 2008.
- [19] M. E. Taylor. *Partial differential equations. I*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996. Basic theory.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907